Analysis of the Dynamics of a Fractal Tree in Wind

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INTRODUCTION

Wind-induced oscillations of a fractal tree are an interesting topic of study because the system contains many nonlinear interactions. For this system, we consider the different levels of the branches and the different oscillations that can occur as a result of sinusoidal wind forces. We examine two nonlinear models to investigate the effect of structural nonlinearities at various nodes (branches) of the fractal tree.

While the influence of various system parameters such as tree's age, taper and slenderness ratio on the tree oscillations would be fantastic things to study, given our limited knowledge of applied methodologies, we focus on a simplified model. Additionally we note that, in its unperturbed state, the branches of the fractal tree in the same branch level are equilateral to one another and are always 60 degrees (π/3 radians) apart from one another. In its perturbed state, the fractal tree will have branches oscillating about this prefered angle.

LITERATURE REVIEW

In 1974, Papesh developed a windthrow model, referring to tree uprooting due to turbulent winds, using the natural frequency v= Aω cos(ωt) where v is the velocity of the wind, ω is the frequency of the wind, and A is the amplitude of the wind gust. He arrived at:

$$A_r = \frac{\pi}{120} c_p c_d \rho \bar{v}^2 AH \frac{\theta^0}{R} + \frac{A_w}{0.625 + \frac{1.891\xi\omega W}{\rho c_d \bar{v} A g}},$$

Through this, Papesh predicted the velocity at which windthrow would occur. However, the model is basic in nature and does not take the crown into account as a separate mass.

In 1994, Gardiner derived this model assuming the tree to be a damped harmonic oscillator with the tree to be a beam with an end mass for the crown while neglecting the mass of the stem. He gave the equation of motion as follows:

$$m_e \frac{\partial^2 y}{\partial t^2} + c\frac{\partial y}{\partial t} + ky = F_o e^{i\omega t}$$

It was found that this model predicted displacement close to the measured values for frequency below the natural frequency of a tree. However, it did not predict accurate responses for frequencies higher than the resonant frequency.

In 1998, Kerzenmacher and Gardiner decided to work together and divide the tree into smaller segments, each with mass, stiffness, and damping parameter. These segments were then joined together to set up a whole system which resulted in a set of differential equations which could be written as follows:

$$\underline{m}\ddot{y} + \underline{c}\dot{y} + \underline{k}y = 0,$$

where m, c, and k are NxN matrices and y is the vector displacement. A transfer function was calculated by solving the equations that were then used to calculate the tree's response when subject to wind forces. This model predicted the deflections well at the top of the tree but failed to do so at lower heights of the tree at frequencies above the resonant frequency of the tree.

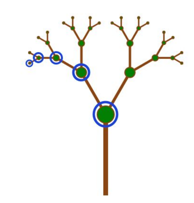
Our analytical approach solves the Kerzenmacher and Gardiner equation as shown in section 10.2 of Kleppner and Kolenkow's *Introduction to Mechanics.* The solution is of the form:

$$x = Ae^{-\frac{b}{m}t}\cos\left(\left[\frac{k}{m} - \frac{b^2}{4m^2}\right]^{\frac{1}{2}} \cdot t\right)$$

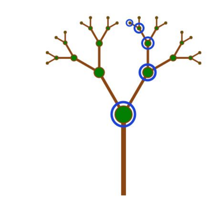
$$\frac{dx}{dt} = -Ae^{-\frac{b}{m}t}\left\{\left[\frac{k}{m} - \frac{b^2}{4m^2}\right]^{\frac{1}{2}} \cdot \sin\left(\left[\frac{k}{m} - \frac{b^2}{4m^2}\right]^{\frac{1}{2}} \cdot t\right) + \frac{b}{2m}\cos\left(\left[\frac{k}{m} - \frac{b^2}{4m^2}\right]^{\frac{1}{2}} \cdot t\right)\right\}$$

Where 'b' is interpreted as the damping coefficient and k is the spring constant.

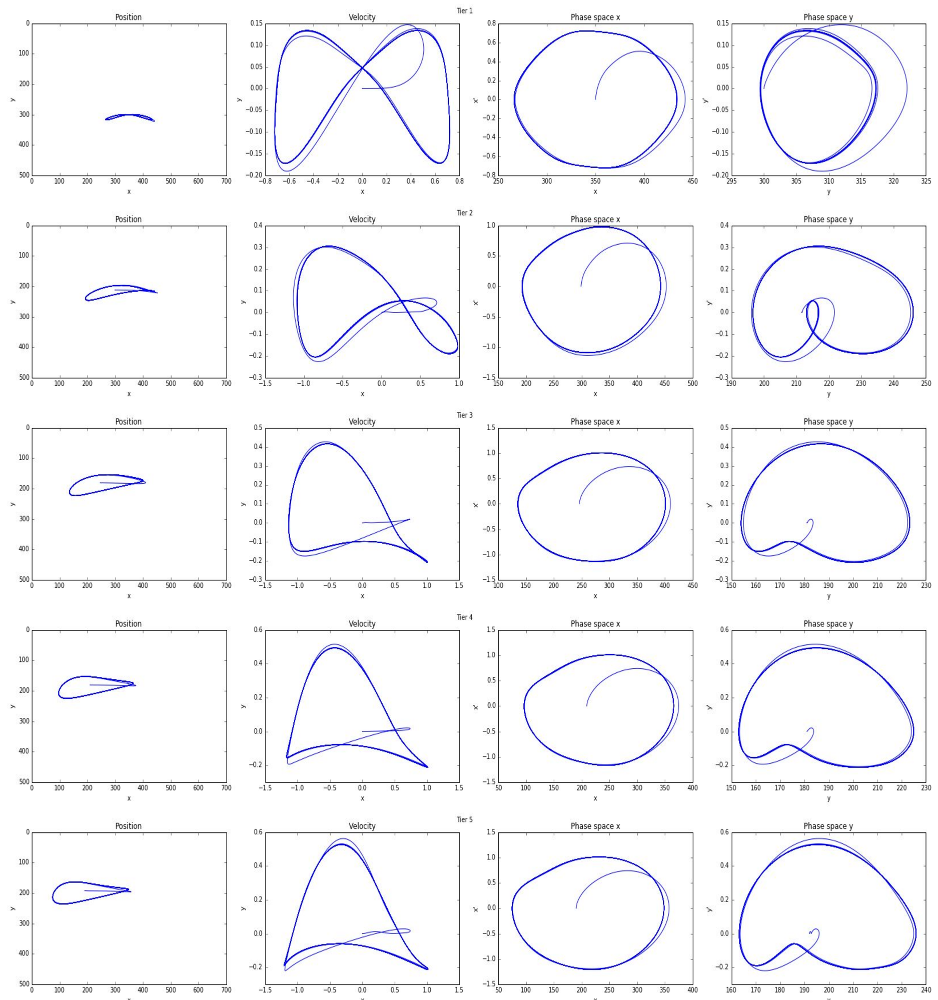
EXPLORING PHASE SPACE



Interactive demo: https://greydanus.github.io/fractal_tree/index.html

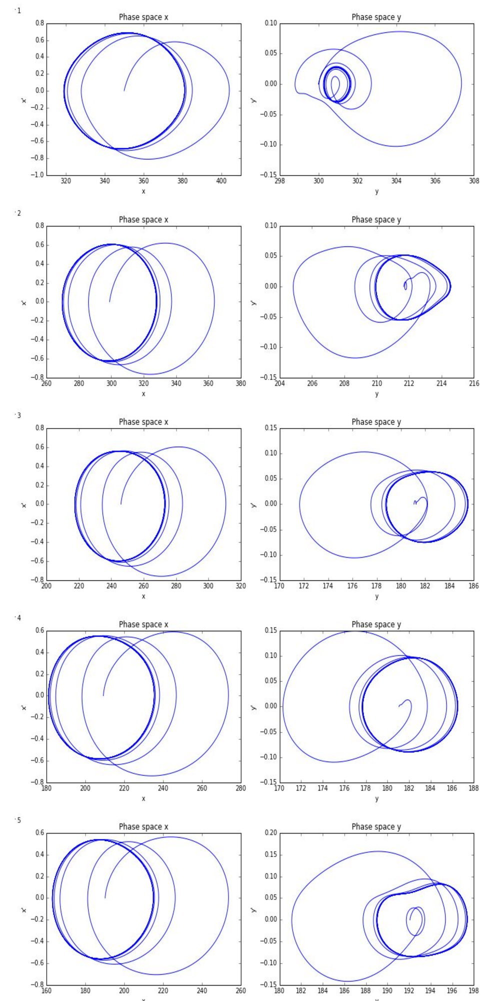


Wavelength: 800 iterations (leftmost nodes)



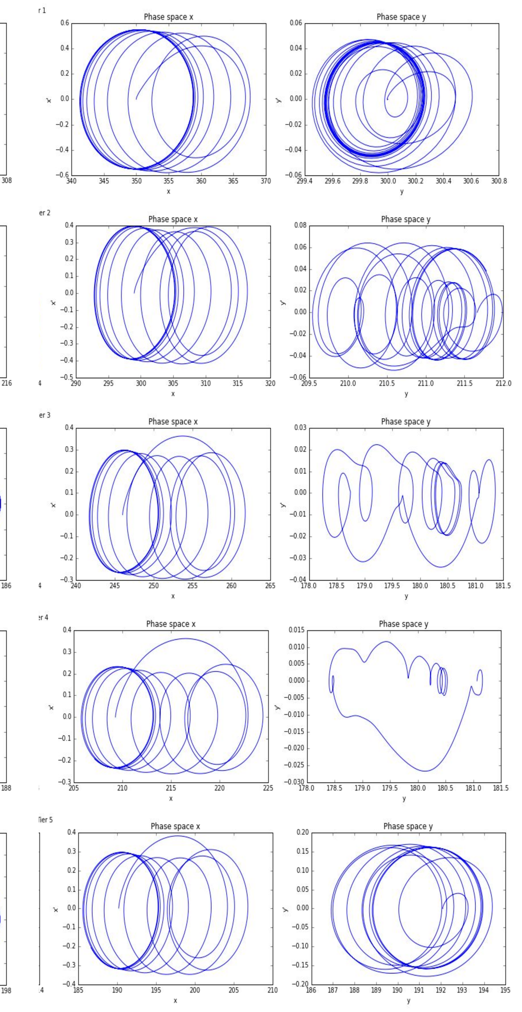
Above: data was collected from the nodes with blue circles (one at each tier) for 10 cycles. This system quickly converges to a periodic cycle. Nodes higher in the tree have more irregular phase spaces

Wavelength: 300 iterations



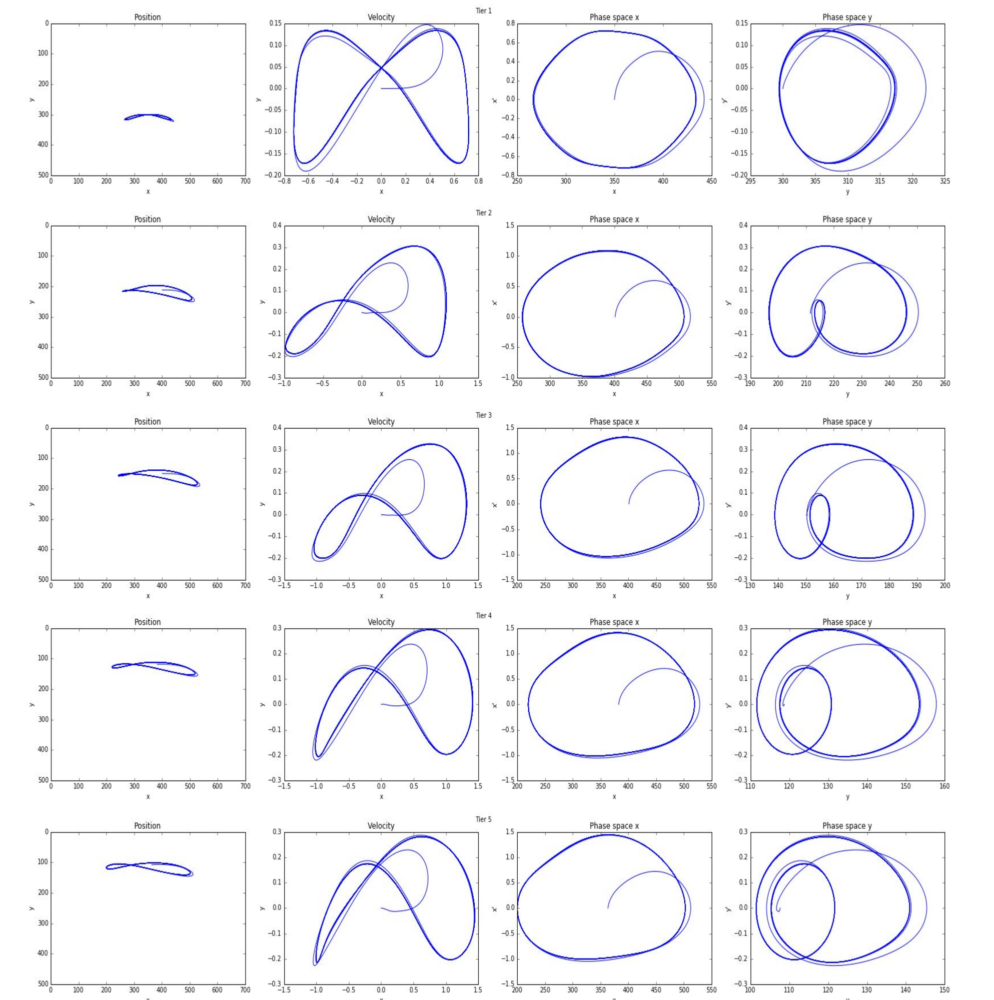
For a shorter wavelength the system takes longer to converge. Also, the periodic cycles vary more by tier

Wavelength: 100 iterations



The system's behavior is most chaotic for very short wavelengths

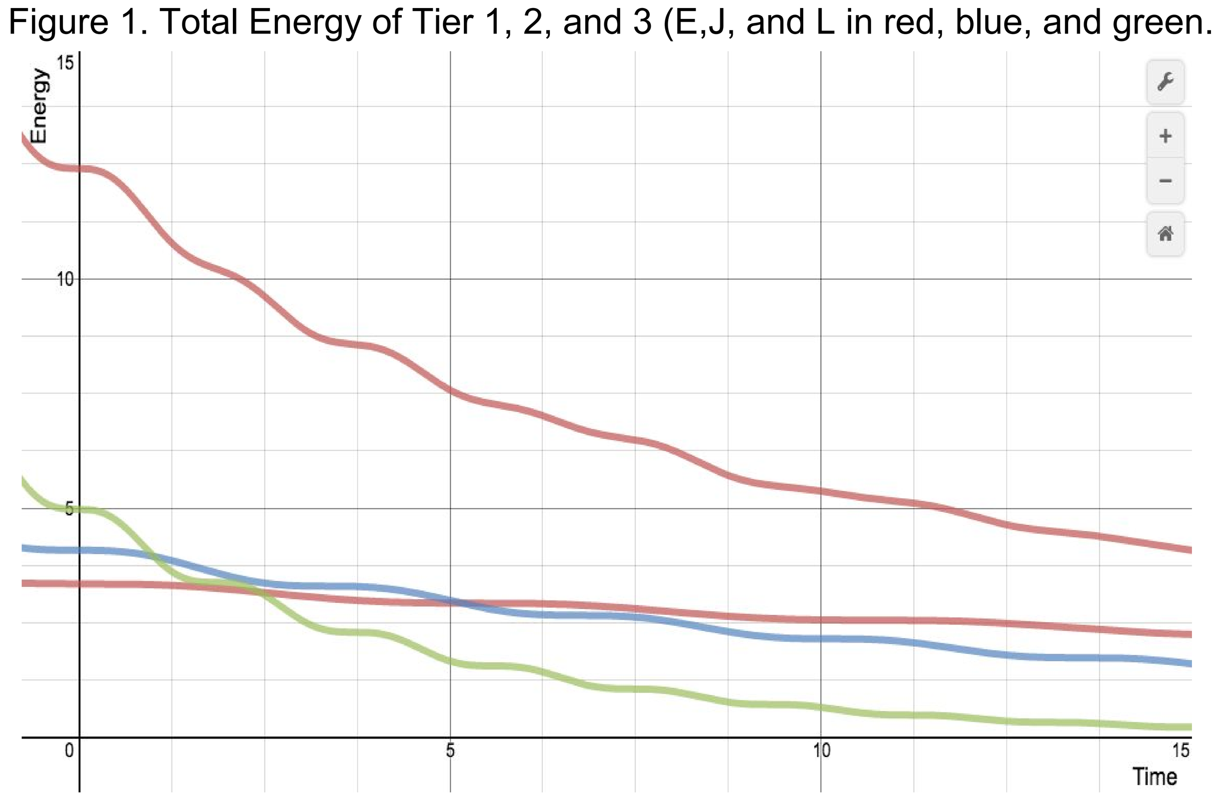
Wavelength=800 (central nodes)



An analysis of the centermost nodes in the tree shows that they have different phase spaces from the leftmost nodes. The inner loop in the y phase space plots is an interesting behavior which emerges at nearly very tier.

EXPLORING ENERGY DISTRIBUTION

Figure 1. Total Energy of Tier 1, 2, and 3 (E,J, and L in red, blue, and green.)



1. $M_{t1} = \left(f^0 m\right) + \left(nf^1 m\right) + \left(n^2 f^2 m\right)$

$M_{t1} = 6.322$

2. $M_{t2} = \left(0 \cdot f^0 m\right) + \left(f^1 m\right) + \left(nf^2 m\right)$

$M_{t2} = 2.436$

3. $M_{t3} = \left(0 \cdot f^0 m\right) + \left(0 \cdot f^1 m\right) + \left(f^2 m\right)$

$M_{t3} = 0.7105$

4. $w_{p0} = \left(\frac{k}{M_{t1}} - \frac{b^2}{4(M_{t1})^2}\right)^{\frac{1}{2}}$

$w_{p0} = 0.748244825729$

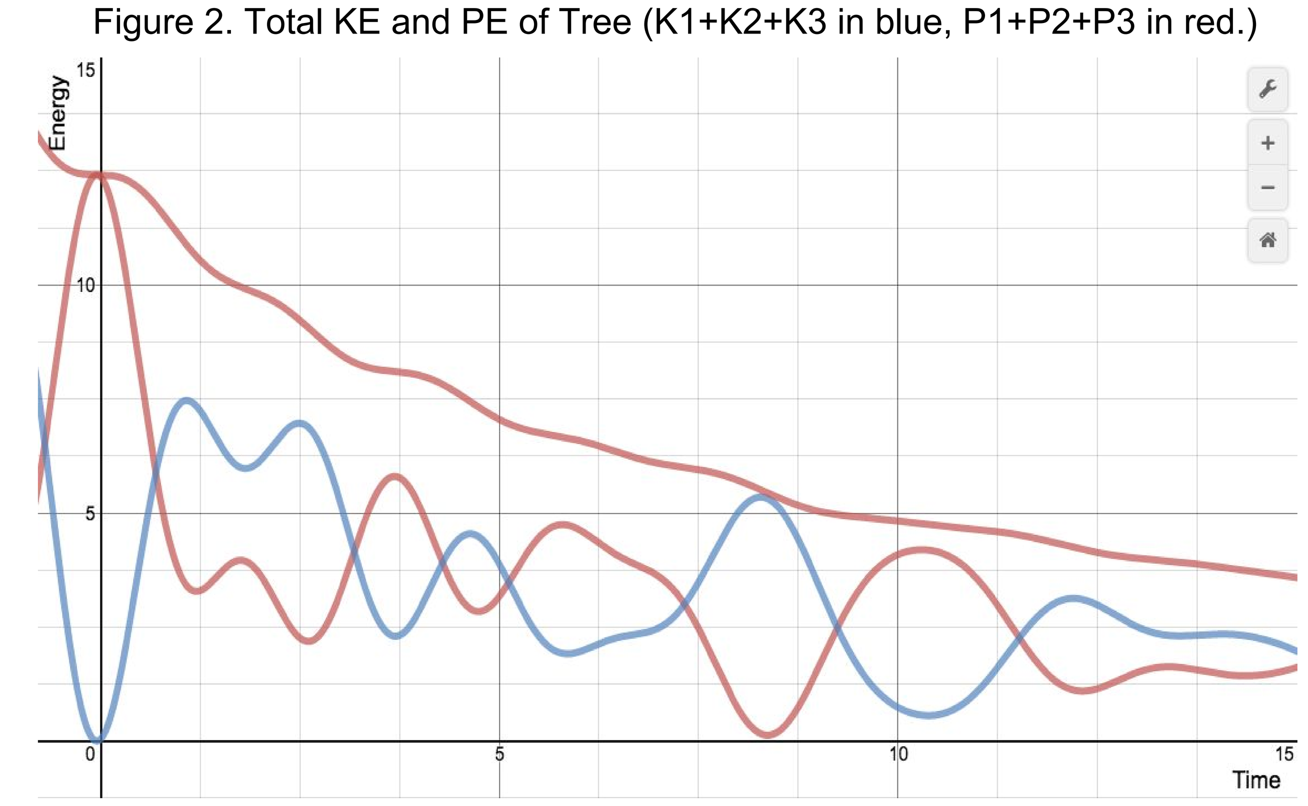
5. $w_{p1} = \left(\frac{k}{M_{t2}} - \frac{b^2}{4(M_{t2})^2}\right)^{\frac{1}{2}}$

$w_{p1} = 1.19982176248$

6. $w_{p2} = \left(\frac{k}{M_{t3}} - \frac{b^2}{4(M_{t3})^2}\right)^{\frac{1}{2}}$

$w_{p2} = 2.18032699812$

Figure 2. Total KE and PE of Tree (K1+K2+K3 in blue, P1+P2+P3 in red.)



7. $E = n^0\left(\frac{1}{2}M_{t1}\left(\left[\left(-f^0 A\right)e^{-\frac{bx}{2M_{t1}}}\left(w_{p0}\sin\left(w_{p0}x\right) + \left(\frac{b}{2M_{t1}}\right)\cos\left(w_{p0}x\right)\right)\right]\right)^2 + \frac{1}{2}k\left(f^0 Ae^{-\frac{bx}{2M_{t1}}}\cos\left(w_{p0}x\right)\right)^2\right)$

8. $J = n^1\left(\frac{1}{2}M_{t2}\left(\left[-\left(f^1 A\right)e^{-\frac{bx}{2M_{t2}}}\left(w_{p1}\sin\left(w_{p1}x\right) + \left(\frac{b}{2M_{t2}}\right)\cos\left(w_{p1}x\right)\right)\right]\right)^2 + \frac{1}{2}k\left(f^1 Ae^{-\frac{bx}{2M_{t2}}}\cos\left(w_{p1}x\right)\right)^2\right)$

9. $L = n^2\left(\frac{1}{2}M_{t3}\left(\left[-\left(f^2 A\right)e^{-\frac{bx}{2M_{t3}}}\left(w_{p2}\sin\left(w_{p2}x\right) + \left(\frac{b}{2M_{t3}}\right)\cos\left(w_{p2}x\right)\right)\right]\right)^2 + \frac{1}{2}k\left(f^2 Ae^{-\frac{bx}{2M_{t3}}}\cos\left(w_{p2}x\right)\right)^2\right)$

14. $K_1 = \frac{1}{2}m\left(\left[\left(-f^0 A\right)e^{-\frac{bx}{2M_{t1}}}\left(w_{p0}\sin\left(w_{p0}x\right) + \left(\frac{b}{2M_{t1}}\right)\cos\left(w_{p0}x\right)\right)\right]\right)^2$

15. $K_2 = \frac{1}{2}nfm\left(\left[\left(-f^1 A\right)e^{-\frac{bx}{2M_{t2}}}\left(w_{p1}\sin\left(w_{p1}x\right) + \left(\frac{b}{2M_{t2}}\right)\cos\left(w_{p1}x\right)\right)\right]\right)^2 + \frac{1}{2}nfm\left(\left[\left(-f^0 A\right)e^{-\frac{bx}{2M_{t1}}}\left(w_{p0}\sin\left(w_{p0}x\right) + \left(\frac{b}{2M_{t1}}\right)\cos\left(w_{p0}x\right)\right)\right]\right)^2$

16. $K_3 = \frac{1}{2}n^2f^2m\left(\left[\left(-f^2 A\right)e^{-\frac{bx}{2M_{t3}}}\left(w_{p2}\sin\left(w_{p2}x\right) + \left(\frac{b}{2M_{t3}}\right)\cos\left(w_{p2}x\right)\right)\right]\right)^2 + \frac{1}{2}n^2f^2m\left(\left[\left(-f^1 A\right)e^{-\frac{bx}{2M_{t2}}}\left(w_{p1}\sin\left(w_{p1}x\right) + \left(\frac{b}{2M_{t2}}\right)\cos\left(w_{p1}x\right)\right)\right]\right)^2 + \frac{1}{2}n^2f^2m\left(\left[\left(-f^0 A\right)e^{-\frac{bx}{2M_{t1}}}\left(w_{p0}\sin\left(w_{p0}x\right) + \left(\frac{b}{2M_{t1}}\right)\cos\left(w_{p0}x\right)\right)\right]\right)^2$

Figure 1 and Figure 2 (as well as the equations beneath them) represent two different methods of modeling energy. Figure 1 plots Eq E, J, and L, which are of the form E=.5mv^2+.5kx^2. These equations represent the PE+KE for each tier. We use M (t1,2,3) as the masses because the effective mass at the end of each branch is the sum of all the mass above it. The x and dx/dt is also calculated using M(t1,2,3). The sum of the three resulting equations is theoretical and in fact actually equal to the sum of the total PE and KE plotted in Figure 2.

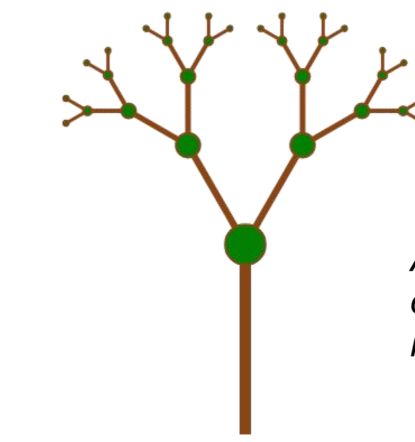
Figure 2 plots Eq K(1,2,3) and P(1,2,3) as well as the total energy. Eq K(1,2,3) are calculated by taking the velocity of the M(t1,2,3) systems and the mass of individual nodes, while Eq P(1,2,3) are calculated the same way as in E, J, and L. In this method we examine the KE of individual nodes in isolation and we can gain a picture of the total KE of the system by summing these nodes. Because each node in K(1,2,3) has KE contributions from more than one tier, cross-comparison of the methods displayed in Figure 1 and Figure 2 is difficult. The KE of E is equal to (1st term K1) + (2nd term K2) + (3rd term K3). The KE of J is (1st term K2) + (2rd term K3). The KE of L is (1st term K3).

17. $P_1 = \frac{1}{2}k\left(f^0 Ae^{-\frac{bx}{2M_{t1}}}\cos\left(w_{p0}x\right)\right)^2$

18. $P_2 = \frac{1}{2}nk\left(f^1 Ae^{-\frac{bx}{2M_{t2}}}\cos\left(w_{p1}x\right)\right)^2$

19. $P_3 = \frac{1}{2}nnk\left(f^2 Ae^{-\frac{bx}{2M_{t3}}}\cos\left(w_{p2}x\right)\right)^2$

MODELING ASSUMPTIONS



An unperturbed snapshot of our JavaScript fractal model

Phase Space (JavaScript model)

1. Tree obeys fractal structure (above)
2. Wind force varies sinusoidally
3. Physical properties diminish by scaling factor f at each tier
   a. Branch length
   b. rotational restoring constant k
   c. Radius
4. Rotational restoring k >> translational restoring k
5. Branches emerge from nodes at branching angles (+/-) π/6
6. Wind force acts on nodes by product with diameter

Energy Distribution

1. Properties 1 and 2 above
2. Branches only experience small oscillations
3. Branching angles are close to vertical

Previous models

1. Assumed tree as cantilever tapered beam
2. End mass assumed as crown of tree (branches + leaves)
3. Model derived was:
4. $$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) + \left(\rho A + \rho_C A_C\right)\frac{\partial^2 y}{\partial t^2} = f$$
   a. with Young's Modulus (*E*), second moment of inertia(*I*), frontal area exposed (A), density of tree (ρ), density of crown (ρc), area of crown (Ac), and force due to wind on the beam per unit length (f)

DISCUSSION

Phase space plots reveal more chaotic behavior for short wavelengths. When cycles in phase space converge, their shapes vary according to location in the tree. We were surprised to see "inner loop" patterns in the phase space plots for most of the center nodes.

For energy analysis, we note that total energy loss in a tier is steepest when KE peaks. A tier experiences no energy loss when PE peaks and KE=0. In Figure 2 we observe the total PE and KE of the tree, as well as the total energy of the tree. Total energy falls faster as KE rises in Figure 2.

Future work could involve extending our model to three dimensions and making a leaf vs no leaf model to compare wind force on a deciduous tree in winter and summer.

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